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Zeroes of the Riemann Zeta Function and Spectral Statistics of Random Matrices

APPM 4360

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April 20th, 2026

Random Matrix Theory

- Random matrices are sampled from a probability distribution.
- The statistical behavior of their eigenvalues are of particular interest.
- Considered by some as a possible approach (**Hilbert-Polya conjecture**) to solving the **Riemann hypothesis** due to similarities between the zero statistics of the zeta function and eigenvalue distributions of some random matrices.

The Riemann Zeta Function

The zeta function is defined as the following Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

which converges absolutely for $\mathrm{Re}(s) > 1$.

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Analytic Continuation

The **gamma function** is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \mathrm{Re}(s) > 0.$$

Key properties:

- Non-zero on $\mathbb{C}$.
- The **reciprocal gamma function** $1/\Gamma(s)$ is entire with simple zeroes for all non-positive integers.

Analytic Continuation cont.

Xi Function

$$\xi(s) := \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}[\theta(u) - 1]du.$$

Apply the functional equation $\theta(u) = t^{-1/2}\theta(1/t)$.

Xi Function, Pole Form

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{(-s/2)-1/2} + u^{(s/2)-1}\right)\psi(u)\,du.$$

We clearly observe simple poles at $s = 0$ and $s = 1$.

Analytic Continuation cont.[3]

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We now solve for the zeta function:

$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}.$$

The pole of $\xi(s)$ for $s = 0$ is canceled out by the corresponding zero of $1/\Gamma(s/2)$.

Analytic Continuation

Therefore, the zeta function has a meromorphic continuation to the entire complex plane with a single simple pole at $s = 1$.

Trivial and Non-Trivial Zeroes

The zeroes of the zeta function corresponding to the zeroes of $1/\Gamma(s/2)$ are considered the **trivial zeroes**. These occur at all negative even integers ($s = -2, -4, -6, \dots$).

All other zeroes of $\zeta(s)$ are considered **non-trivial zeroes.**

Riemann Hypothesis (1859)

All non-trivial zeros of the zeta function lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

It is only proven that non-trivial zeros are restricted to the critical strip $0 < \mathrm{Re}(s) < 1$.

The Critical Strip, $\mathrm{Re}(s) > 1$

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Euler Product Formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad \mathrm{Re}(s) > 1$$

$\zeta(s)$ converges absolutely for $\mathrm{Re}(s) > 1$ and is a product of nonzero terms. Therefore, it is non-zero on this region.

The Critical Strip, $\mathrm{Re}(s) < 0$

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Functional Equation for $\zeta(s)$

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

- $\zeta(1-s)$ is nonzero because $\mathrm{Re}(1-s) > 1$.
- $\Gamma((1-s)/2)$ is nonzero.

Therefore, the only zeroes for $\mathrm{Re}(s) < 0$ are the trivial zeroes.

Zeroes on the Critical Line, $\mathrm{Re}(s) = \frac{1}{2}$

The **critical line** is defined when $s = \frac{1}{2} + it$. We are interested in finding the zeroes γ_n on this line when $\zeta(\gamma_n = \frac{1}{2} + it_n) = 0$.

In practice, one can extract a function from the log-transform of $\xi(\frac{1}{2} + it)$ that is always real-valued on $s = \frac{1}{2} + it$ and defined in terms of t called the **Z function**:

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$$
$$= e^{i\left(\mathbf{Im}\log\Gamma\left(\frac{1}{4}+i\frac{1}{2}t\right)-\frac{1}{2}t\log\pi\right)}\zeta\left(\frac{1}{2} + it\right)$$

- $Z(t) \in \mathbb{R}, \quad s = \frac{1}{2} + it$
- $Z(t_n) = 0 \leftrightarrow \zeta\left(\frac{1}{2} + it_n\right) = 0$

Zeroes on the Critical Line: Z Function cont.

Riemann-Siegel approximation of $Z(t)$

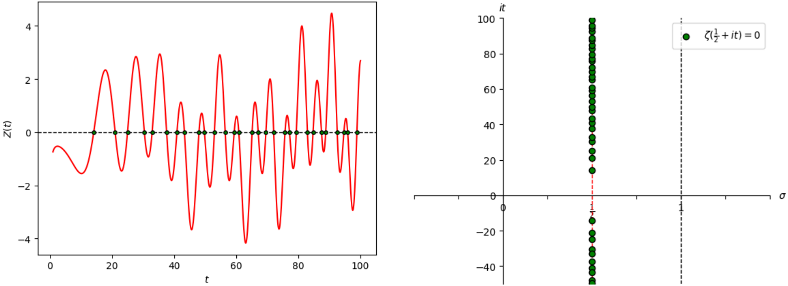
$$Z(t) \approx 2 \sum_{n=1}^{\text{floor}\sqrt{\frac{t}{2\pi}}} \frac{1}{\sqrt{n}} \cos\left(t \log n - \theta(t)\right) + R(t)$$

$$\theta(t) = \mathbf{Im} \log \Gamma\left(\tfrac{1}{4} + i\tfrac{1}{2}t\right) - \tfrac{1}{2} t \log \pi$$

- Numeric estimate of $Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$.
- Derived from an integral representation of $\zeta(s)$ (Edwards 1974).
- To find t_n s.t. $\zeta\left(\frac{1}{2} + it_n\right) = 0$: Iterate through $Z(t)$ to $t = T$ and use root-finding between sign changes.

Zeroes on the Critical Line: Example

Riemann-Siegel with first remainder term, $t \in [1, 100]$:



We will look at the normalized pairwise distances of these zeroes and their relationship to the eigenvalues of random matrices from certain ensembles.

Random Matrix Overview

Random matrices have elements that are random variables. They are identified with **ensembles** which are probability distributions over matrices.

Examples:

- Gaussian Unitary Ensemble (GUE)
- Gaussian Orthogonal Ensemble (GOE)
- Circular Unitary Ensemble (CUE)

Applications such as modeling the distance between energy levels in atomic nuclei with eigenvalue spacings.

We will compare the spectral spacing distributions of the GUE and CUE to those of $\zeta\left(\frac{1}{2} + it\right)$.

GUE Definition

Gaussian Unitary Ensemble (GUE)

Set of $N \times N$ random Hermitian ($H = \overline{H}^T$) matrices with Gaussian probability distribution, which does not change when multiplied by a fixed Unitary matrix.

Key properties:

- Eigenvalues are all real ($\lambda_j \in \mathbb{R}$).
- Eigenvalues have probability density
  $$p(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,2}} \prod_{1 \le j < k \le n} \left|\lambda_j - \lambda_k\right|^2 e^{-\sum_{j=1}^n \lambda_j^2}.$$
- Can be constructed as follows:
  $$H = \frac{A + \overline{A}^T}{2}, \quad A_{jk} = X_{jk} + iY_{jk}, \quad X_{jk}, Y_{jk} \sim N(0,1)$$

CUE Definition

Circular Unitary Ensemble (CUE)

Set of $N \times N$ random Unitary ($U\overline{U}^T = I$) matrices whose uniform probability distribution does not change when it is multiplied by another fixed Unitary matrix.

Key properties:

- Eigenvalues are all on the complex unit circle ($\lambda_j = e^{i\theta_j}$).
- Eigenvalue angles (**eigenangles**) have probability density $p(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,2}} \prod_{1 \le j < k \le n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2$.

The Keating-Snaith Correspondence

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- **Thesis:** The "moments" or statistical descriptors (mean, variance, etc.) of the zeta function's values on the critical line are modeled by the characteristic polynomials of CUE matrices.
- **Density Mapping:** The mean density of the zeta zeros must be equal to the mean density of the matrix eigenvalues:

$$N = \log\left(\frac{T}{2\pi}\right)$$

- **Synthesis:** By calculating the asymptotic limit of the CUE moments, the Keating-Snaith Correspondence predicts the coefficients for the growth of zeta moments, indicating asymptotic convergence between zeta and CUE statistics.

Eigenvalue/Eigenangle Repulsion

GUE Eigenvalue Joint PDF

$$p\left(\lambda_1, \ldots, \lambda_n\right) = \frac{1}{Z_{n,2}} \prod_{1 \leq j < k \leq n} \left|\lambda_j - \lambda_k\right|^2 e^{-\sum_{j=1}^{n} \lambda_j^2}$$

CUE Eigenangle Joint PDF

$$p(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,2}} \prod_{1 \leq j < k \leq n} \left|e^{i\theta_j} - e^{i\theta_k}\right|^2$$

The product term $|a - b|^2$ forces points to be far apart, i.e. **repulsion**.

Pair Correlation Function

$R_2(r)$

Given a zero at any position, $R_2(r)$ is the density of zeroes that are at a normalized distance r away on the critical line.

Notes:

- For uniformly distributed random points, $R_2(r) = 1$.

Montgomery's Pair Correlation Conjecture

PCC

Assuming the Riemann Hypothesis,

$$R_2(r) = 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2.$$

Notes:

- $R_2(r) \to 0$ as $r \to 0$, indicating **repulsion.**
- Derived using Fourier spectrum analysis and prime number theory.
- Not a universally proven theorem due to restriction on frequency domain.

Contour Integration Argument

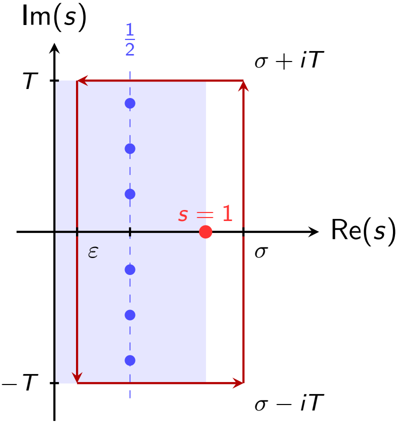
We will use the **logarithmic derivative** of $\zeta(s)$ given by $\frac{\zeta'(s)}{\zeta(s)}$.

Properties:

- Simple pole for every non-trivial zero z_n.
- Each pole has residue $+1$.

Contour Integration Argument cont.[2]

Define C as a rectangular closed contour whose vertices are $\varepsilon \pm iT$ and $\sigma \pm iT$ for $\sigma = 1 + \varepsilon$ for a small $\varepsilon > 0$ ($T \to \infty$):



Contour Integration Argument cont.[3]

Argument Principle (Ablowitz 4.4.5)

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} h(z)dz = \sum_{i_z=1}^{M_z} n_{iz} h(z_{iz}) - \sum_{i_p=1}^{M_p} n_{ip} h(z_{ip})$$

We integrate the logarithmic derivative over C with a test function g and apply the argument principle to get:

$$\frac{1}{2\pi i}\oint_C g(s)\frac{\zeta'(s)}{\zeta(s)} ds = \sum_{i=1}^{M_z} g(z_i) - g(1)$$

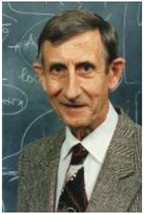
Contour Integration Argument cont.[4]

Montgomery (1987) chooses a test function g such that:

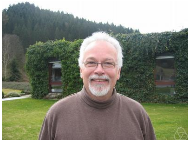
- The top and bottom sides of the contour contribute $O(T^{-1}\log^2 T)$ and thus vanish as $T \to \infty$.
- The left side evaluates to a manageable real quantity.
- The right side becomes a weighted prime sum that is handled using the prime number theorem.

Montgomery then uses the equation for the sum over the zeros to derive $R_2(r)$.

Dyson-Montgomery Correspondence



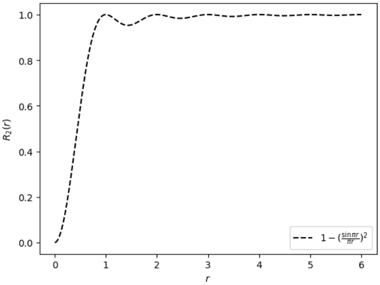
Freeman Dyson



Hugh Montgomery

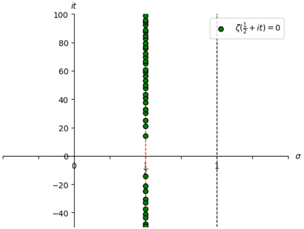
Freeman Dyson noticed that Montgomery's correlation function $R_2(r) = 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2$ for $\zeta\left(\frac{1}{2} + it\right)$ matches those of the random matrices he studied, such as the GUE and CUE.

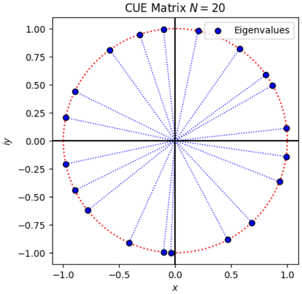
Pair Correlation Statistics



Given that Riemann Hypothesis is true, then the Montgomery conjecture suggests that the occurrences of two zeroes with distance r on $\frac{1}{2}+it$ is modeled by $R_2(r)$.

Pair Correlation Statistics cont.

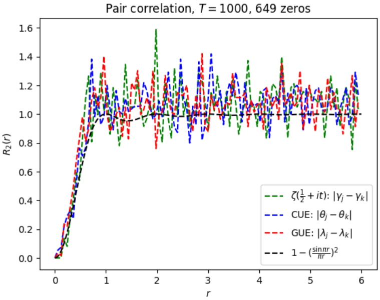




To find pair correlation:

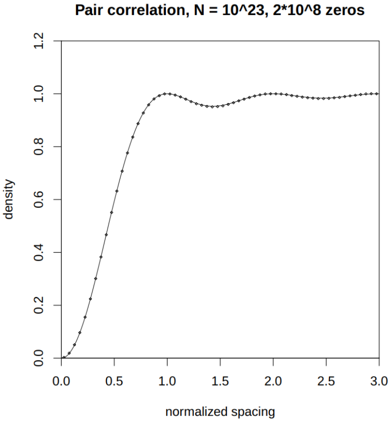
- For each pair of points p_j, p_k, find $\left|p_j - p_k\right|$.
- Normalize the distances: $r = \frac{\left|p_j - p_k\right|}{\text{mean}\left(\left|p_j - p_{j+1}\right|\right)}$
- Count frequency of each r (histogram).
- Normalize histogram by dividing by mean frequency on interval.

Pair Correlation Statistics cont.



Indeed, the eigenvalue/eigenangle and $\zeta\left(\frac{1}{2} + it\right)$ zero pairwise distances do appear to exhibit the same pair correlation statistics, suggesting a connection between $\zeta(s)$ and random matrices.

Odlyzko Pair Correlation Data



"Figure 7.1: Pair correlations of (spacing-normalized) zeta zeroes around height 10^{23}, and the random matrix prediction. Figure courtesy of Andrew Odlyzko."

Source: Elizabeth S. Meckes, *The Random Matrix Theory of the Classical Compact Groups*, pg. 198.

Implications for the Riemann Hypothesis

The **Riemann Hypothesis** states that all non-trivial zeroes of $\zeta(s)$ lie on $s = \frac{1}{2} + it$.

According to the **Hilbert-Polya conjecture**, if a Hermitian operator exists such that its eigenvalues correspond to the non-trivial zeroes, then **it would imply that the Riemann-Hypothesis is true**.

GUE matrices are Hermitian. However, only their eigenvalue statistics are similar to $\zeta(s)$, and **no Hermitian operator that satisfies the above conjecture has been found**.

Conclusion

- While definite proofs of the connection between random matrices and the zeta function are limited, extensive numerical work demonstrates an undeniable link between the two fields.
- CUE matrices seem to model the distribution of $\zeta(\frac{1}{2} + it)$ more closely than GUE in some respects.
- The behavior of the zeta function is still heavily related to prime number theory.

Discussion

Thank you!

Questions?