Zeros of the Riemann Zeta Function and Spectral Statistics of Random Matrices

APPM 4360
Section 001
Complex Variables and Applications

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Contents

1 Abstract

This report investigates the Riemann zeta function's special properties concerning its zeros and their connection to the eigenvalue distributions of random matrices. Complex analytic methods are applied to establish the locations of the trivial and non-trivial zeros within the critical strip, and to motivate the Riemann Hypothesis. A numerical implementation locating zeros along the critical line via the Riemann-Siegel formula is presented, and computational results reveal repulsion between zeros consistent with the Montgomery pair correlation conjecture. The Circular Unitary Ensemble is introduced as a random matrix model whose eigenvalue spacing statistics mirror those of the zeta zeros, a correspondence supported both numerically and through the Keating-Snaith conjecture, which yields analytically verifiable predictions for the moments of the zeta function along the critical line. The report concludes with a discussion of the implications of this correspondence for the Hilbert-Pólya conjecture and the broader program of approaching the Riemann Hypothesis through random matrix theory.

2 Introduction

The Riemann zeta function is a key function in number theory that has been thoroughly studied using complex analysis tools, most notably analytic continuation. In this section, we extend the zeta function to a meromorphic function on the entire complex plane and discuss its fundamental properties such as poles and zeros.

The zeta function is defined as the following Dirichlet series [13, p. 168]:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1. \tag{1}$$

Note that absolute convergence for $\mathrm{Re}(s) > 1$ follows from the modulus of the summand taking the form of a real p-series.

The study of the zeta function is closely related to the theory of prime numbers, with a notable form of the zeta function being the Euler product formula defined by [13, p. 182]

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \mathrm{Re}(s) > 1. \tag{2}$$

which is also absolutely convergent on $\mathrm{Re}(s) > 1$.

To begin the analytic continuation of ζ, we recall the gamma function, defined by [13, p. 161]

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \mathrm{Re}(s) > 0. \tag{3}$$

The reciprocal gamma function $1/\Gamma(s)$ is entire with simple zeros for all non-positive integers. The gamma function itself is non-zero on the entire complex plane [13, p. 161].

Using the gamma function and a real series known as the Jacobi theta function, one can construct the xi function [13, p. 170]

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{4}$$

which satisfies the functional equation [13, p. 170]

$$\xi(s) = \xi(1-s) \tag{5}$$

and has simple poles for $s = 0$ and $s = 1$.

We now demonstrate that the zeta function has a meromorphic continuation into the entire complex plane by rearranging (4) to get [13, p. 172]

$$\zeta(s) = \pi^{s/2} \frac{\xi(s)}{\Gamma(s/2)}.$$

Note that $1/\Gamma(s/2)$ has simple zeros for $s = 0, -2, -4, \ldots$, meaning the simple pole of $\xi(s)$ at $s = 0$ is canceled out by the zeros of $1/\Gamma(s/2)$, with $\zeta(s)$ approaching a finite nonzero value for this point given that the pole and zero are of the same order [13, p. 172].

Therefore, the zeta function has a meromorphic continuation to the entire complex plane with a single simple pole at $s = 1$.

The zeros of $\zeta(s)$ corresponding to the zeros of $1/\Gamma(s/2)$ at $s = -2, -4, -6, \ldots$ are known as the function's trivial zeros. Of greater interest are the non-trivial zeros of the zeta function, which were conjectured to exist only on the critical line $\mathrm{Re}(s) = 1/2$ by Bernhard Riemann in 1859, defining the still-unsolved Riemann Hypothesis. While an explicit proof for the conjecture has evaded mathematicians for nearly two centuries, our report seeks to express that the statistical behaviors of the non-trivial zeros exhibit peculiar similarities with the statistics of objects from a field known as random matrix theory.

Random matrix theory (RMT) is concerned with matrices sampled from specific probability distributions, whose elements are random variables. It studies the statistical distributions of resulting eigenvalues for various matrix ensembles, defined as sets of random matrices with rigorous probability distributions defined over them [9].

A canonical choice of ensemble when discussing the zeta function is the Gaussian Unitary Ensemble (GUE), defined as the set of square Hermitian matrices H, $H = H^\dagger$, with a probability distribution that is invariant under transformation by a unitary matrix $H \overset{d}{=} UHU^\dagger$, hence the terminology "unitary" when referring to the ensemble of Hermitian matrices. The elements of GUE matrices are constructed using samples from the standard Gaussian distribution $\mathcal{N}(0,1)$ [9].

The focus of RMT-zeta study on GUEs throughout the 20th century can be traced back to the Hilbert-Pólya conjecture of 1914, which suggests that the non-trivial zeros of the zeta function can be given by the eigenvalues of some self-adjoint, or Hermitian operator [11][12].

In this report, we work with the Circular Unitary Ensemble (CUE), which contains unitary matrices whose eigenvalues are distributed on the complex unit circle. We focus on the CUE more extensively than the GUE because more recent RMT-zeta work demonstrates a statistical CUE-zeta correspondence grounded in analytically proven cases for which GUE-zeta study lacks a counterpart [7]. We present evidence that the CUE is a compelling object to approach the Riemann Hypothesis despite its matrices lacking a Hermitian property. We also perform numerical analysis to demonstrate both the GUE and CUE's striking statistical correspondences with the zeta function.

3 Non-Trivial Zeros of the Riemann Zeta Function

3.1 The Critical Strip

It can be proven that the non-trivial zeros of the zeta function can only be found on the critical strip $0 < \mathrm{Re}(s) < 1$.

We begin by recalling that $\zeta(s)$ was shown to converge absolutely for $\mathrm{Re}(s) > 1$ in (1). We further note that in the Euler product formula we observe $\zeta(s)$ being an absolutely convergent product of non-zero terms. This implies that $\zeta(s)$ is non-zero for $\mathrm{Re}(s) > 1$.

We now look at $\mathrm{Re}(s) < 0$. We apply the functional equation $\xi(s) = \xi(s-1)$ to (4) and solve for $\zeta(s)$ to express it in the functional form [13, p. 184]

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s) \tag{6}$$

We observe that $\zeta(1-s)$ has no zeros because $\mathrm{Re}(1-s) > 1$ in this region and we have already shown $\zeta(s) \neq 0$ for all $\mathrm{Re}(s) > 1$. We also note that $\Gamma((1-s)/2)$ is non-zero. This leaves only $1/\Gamma(s/2)$, which we have already discussed as causing only the trivial zeros.

Therefore, we have shown that the zeta function has no zeros for $\mathrm{Re}(s) > 1$ and only trivial zeros for $\mathrm{Re}(s) < 0$, making the region $0 < \mathrm{Re}(s) < 1$ the critical strip on which non-trivial zeros may occur.

3.2 Zeros On the Critical Line: The Riemann-Siegel Formula

The non-trivial zeros we will compare with the eigenvalues of the random matrices of the GUE and CUE are on the critical line $s = \frac{1}{2} + it$. A practical way to find these zeros is with the Z function [5,

p. 119]:

$$
\begin{aligned}
Z(t) &= e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right) \qquad (7)\\
&= e^{i\left(\mathbf{Im}\log\Gamma\left(\frac{1}{4}+i\frac{1}{2}t\right)-\frac{1}{2}t\log\pi\right)}\zeta\left(\frac{1}{2}+it\right).
\end{aligned}
$$

The Z function can be shown to always be real on $s=\frac{1}{2}+it$ since it is derived by separating the real and imaginary parts of the log-transform of $\xi\left(\frac{1}{2}+it\right)$, where $1-s=\overline{s}$ and $\xi(s)=\xi(1-s)=\overline{\xi(s)}$ on the critical line. Furthermore, one can see that given

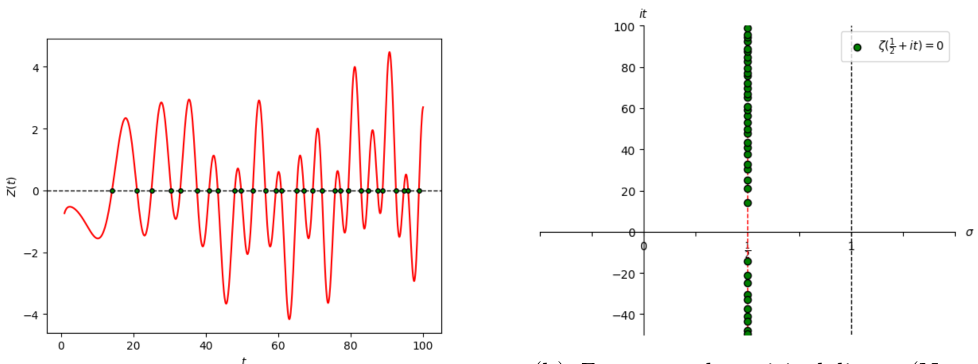
$$
\xi\left(\frac{1}{2}+it\right)=\pi^{-\frac{\frac{1}{2}+it}{2}}\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)\zeta\left(\frac{1}{2}+it\right), \qquad (8)
$$

whenever $\xi=0$, then $\zeta=0$.

A standard way to numerically evaluate the Z function is with the Riemann-Siegel formula, which is derived with steepest-descent from an integral representation of ζ [5, p. 139][12, p. 296]:

$$
Z(t)\approx 2\sum_{n=1}^{\operatorname{floor}\sqrt{\frac{t}{2\pi}}}\frac{1}{\sqrt{n}}\cos\left(t\log n-\theta(t)\right)+R(t). \qquad (9)
$$

In this example, we used the first remainder term for $R(t)$ obtained from [2]



(a) Zeros of $Z(t)$.

(b) Zeros on the critical line. (Note: Zeros are $\pm t_n$.)

Figure 1: Zeros of $\zeta\left(\frac{1}{2}+it\right)$ up to height $t=100$.

The zeros are found by evaluating $Z(t)$ along an interval and using a root-finding algorithm (e.g. bisection method) at points where $Z(t)$ changes sign. We will look at the normalized pairwise distances of these zeros and their relationship to the eigenvalues of our random matrices.

4 Circular Unitary Ensemble

The Circular Unitary Ensemble is defined as a set of $N\times N$ random unitary matrices such that $U^{\dagger}U=I$. Each matrix is distributed according to the Haar measure, defined as the unique probability measure on the unitary group $U(N)$ that is invariant under left and right multiplication by any fixed unitary matrix [8, p. 11]. An important characteristic of the CUE matrices is that their eigenvalues lie on the unit circle in the complex plane. This gives the benefit of rotational symmetry, meaning that the statistical properties of the eigenvalues are invariant for the rotations of the unit circle. This rotational invariance constrains the joint distribution of eigenphases to take the specific form discussed in the following section, where the group's symmetry directly dictates the pattern of the angles [9].

The eigenvalues of CUE matrices can be written as $\lambda_i = e^{i\theta}, \quad \theta_i \in [0, 2\pi)$. This form allows us to reduce the analysis of the eigenvalues to studying the real valued angles, θ_i. The key structure of the system to study is the spacing between the eigenvalue pairs $\theta_{i+1} - \theta_i$.

The resulting eigenphase spacing statistics will be shown to be strikingly similar to those of the zeros of the zeta function, as we will now motivate.

4.1 Eigenvalue Distribution

The characteristic polynomial of a random unitary matrix U from the CUE follows as [7, p. 1]:

$$Z(U, \theta) = det(I - U^{-i\theta}) \tag{10}$$

with the roots of this polynomial being the eigenvalues, or eigenphases, of the matrix. An important property we will explore is the repulsive behavior these eigenvalues exhibit, which is one of the key characteristics of why the CUE eigenvalue distribution can be used to model the zeros of the zeta function. To make this explicit, we rewrite $Z(U, \theta)$ in terms of its eigenvalues $e^{i\theta_n}$ [7, p. 62]:

$$Z(U, \theta) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}) \tag{11}$$

Let $\phi = (\theta_n - \theta)$. To show the magnitude of each factor, we apply Euler's identity:

$$\begin{aligned} \left|1 - e^{i\phi}\right| = \left|e^{-i\frac{\phi}{2}}\right|\left|e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}}\right| &= (1)\left|e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}}\right| \\ &= \left|-2i\sin\left(\frac{\phi}{2}\right)\right| \\ &= 2\left|\sin\left(\frac{\phi}{2}\right)\right| \end{aligned} \tag{12}$$

Therefore, we get:

$$|Z(U, \theta)| = \prod_{n=1}^{N} 2\left|\sin\left(\frac{\theta_n - \theta}{2}\right)\right|. \tag{13}$$

This form shows that the characteristic polynomial has roots whenever $\theta_n = \theta$, i.e. at the eigenphases of the matrix.

Expanding on how the eigenvalues are repulsive to each other, we use the joint probability density function for the eigenphases [7, p. 62]:

$$P(\theta_1, \theta_2, ..., \theta_N) = ((2\pi)^N N!)^{-1} \prod_{j<m} |e^{i\theta_j} - e^{i\theta_m}|^2. \tag{14}$$

Note that the Vandermonde-like factor $\left|e^{i\theta_j} - e^{i\theta_m}\right|^2$ vanishes whenever two eigenphases coincide, demonstrating that the probability of two eigenvalues being close together is extremely unlikely. Thus, the eigenvalues are said to repel each other, or tend to be positioned away from each other.

Before relating these properties to the moments of the zeta function, it is important to note that to accurately model the zeta function higher up on the critical line $\frac{1}{2} + it$, increasingly larger matrices are required. This is due to the density of the zeros on this line being given by the equation $\frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right)$ for a height T up on the critical line [7, p. 72]. This density is derived from the zero counting function $N(T) \sim \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$, whose differentiation yields $\frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right)$. $N(T)$ itself is derived through a contour integration argument similar to that which we apply in Section 6.1 [3, pp. 97-100].

For a CUE matrix of size N, there are N eigenvalues along the unit circle with a circumference of 2π, giving the eigenvalues a mean density of $\frac{N}{2\pi}$. Since our goal is to use the eigenvalues of the CUE matrix to model the zeta function, the mean densities of the two should be equal, giving us the approximation [7, p. 59]:

$$N \approx \log\left(\frac{T}{2\pi}\right) \tag{15}$$

which states that to model a height T up the critical line, a matrix of size N is required.

Just as $|\zeta(1/2+it)|$ vanishes at the non-trivial zeros, $|Z(U,\theta)|$ vanishes at its eigenphases, and in fact the matching of densities allows us to additionally observe that the statistical distribution of eigenphase spacings seems to mimic that of zeta zero spacings, revealing the central motivation for CUE as a zeta model.

5 Similarities in Moments

In this section, we use the concept of moments as they are treated in statistics, i.e. as quantities that describe the properties of a statistical distribution such as shape, center, spread, etc.

5.1 Moments of CUE

The moments of the CUE are given by the following equation [7, p. 59]:

$$M_N(s) = \langle |Z(U,\theta)|^s \rangle \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+\frac{s}{2}))^2} \tag{16}$$

This function gives the average values of its characteristic polynomial when raised to the power s. As before, $Z(U,\theta)$ is the characteristic polynomial of the unitary matrices that increases in magnitude when the eigenvalues are spread farther apart, and goes to zero when the eigenvalues are close together. The magnitude of the polynomial is raised to the power s, which is the 2λ-th moment of the function. Higher order λ values weigh the tails of the distribution more heavily.

As the approximation $N \approx \log(t/2\pi)$ suggests, as the distance up the critical line increases ($T \to \infty$), the size of the matrix must increase ($N \to \infty$). At these very large matrix sizes, the statistical distribution of $\log(Z(U,\theta))$'s values follow a Gaussian distribution. More precisely, the real and imaginary parts of $\log(Z(U,\theta))/\sqrt{(1/2)\log N}$ each separately converge to a Gaussian with zero mean and unit variance as $N \to \infty$ [7, p. 59]. Here $\log(Z) = \log|Z| + i\arg(Z)$ characterizes the Gaussian, where $\log|Z|$ is the log magnitude or height of the characteristic polynomial, and also gives a log normal distribution. The imaginary part $i\arg(Z)$ represents how the function rotates about the origin.

The variance of $\log(Z(U,\theta))$ is $\frac{1}{2}\log(N)$, meaning the s^{th} moment of the underlying log-normal distribution scales as [7, p. 60]:

$$\exp\left(\frac{s^2}{2}\cdot\frac{1}{2}\log(N)\right) = N^{\frac{s^2}{4}}. \tag{17}$$

Using the change of variable $s = 2\lambda$, where lambda is any real or imaginary number, the moments grow at a rate of N^{λ^2}.

This factor is used when considering the limit of the expectation value of $|Z|^{2\lambda}$ rescaled by dividing by the growth factor N^{λ^2}, given by the equation [7, p. 60]:

$$f_{CUE}(\lambda) = \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int \langle |Z(U,\theta)|^{2\lambda} \rangle_{U(N)} \tag{18}$$

$$= \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \text{ for } k \geq 1 \tag{19}$$

where the integral is taken over the entire unitary group $U(N)$, giving a unitary average.

This quantity will be shown to be crucial in the link between CUE and zeta statistics. Conceptually, this is the geometrical representation of the eigenvalue repulsion. As the size of the matrices goes to infinity, the moments of the characteristic polynomial grow at a log-normal growth rate of N^{λ^2}, but when divided by this growth factor, f_{CUE} describes the local interactions between eigenvalues.

5.2 Moments of the Zeta Function

We now analyze the moments of the zeta function. The λ-th moment along the critical line is defined as [14, p. 115]:

$$I_\lambda(T) = \frac{1}{T} \int_1^T \left|\zeta(\frac{1}{2} + it)\right|^{2\lambda} dt \tag{20}$$

The behavior of $I_k(T)$ for all k collectively describes how the values of the zeta function are distributed along the critical line. Specifically, observing the asymptotic behavior of each moment as $T \to \infty$ is equivalent to determining the average distribution of the zeta function along the critical line.

The asymptotic behavior of the first moment was proven by G.H. Hardy and J.E. Littlewood in 1918 as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta(\frac{1}{2} + it)\right|^2 dt \sim \log T \tag{21}$$

[14, p. 118]. Likewise, in 1926, A.E. Ingham proved for the second moment [14, p. 125]:

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \left|\zeta(\frac{1}{2} + it)\right|^4 dt \sim \frac{\log^4 T}{2\pi^2} \tag{22}$$

$\lambda = 1$ and $\lambda = 2$ are the only proven cases. However, these results establish a suspected pattern of $O((\log T)^{\lambda^2})$ growth with an unknown constant, leading Keating and Snaith to make a conjecture for $\lambda \geq 3$.

5.3 The Keating-Snaith Correspondence

Keating and Snaith (2000) conjecture that [7, p. 59]

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta(\frac{1}{2} + it)\right|^{2\lambda} dt \sim (\log T)^{\lambda^2} f(\lambda) a(\lambda) \tag{23}$$

where $a(\lambda)$ is referred to as an arithmetic factor arising purely from number theory defined as [7, p. 59]

$$a(\lambda) = \prod_{p \text{ prime}} \left\{ (1 - 1/p)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda + m)}{m!\,\Gamma(\lambda)} \right)^2 p^{-m} \right) \right\}, \tag{24}$$

and $f(\lambda)$ is claimed to be as follows [7, p. 61]:

$$f(\lambda) = f_{\mathrm{CUE}}(\lambda) \tag{25}$$

This equation introduces the main analytical connection between the statistics of CUE matrices and the zeros of the zeta function along the critical line.

We observe that this conjecture is consistent with the proven cases for $k = 1$ and $k = 2$. For $\lambda = 1$, $f_{\mathrm{CUE}}(1) = 1$ and we also have $a(1) = 1$, recovering $I_1(T) \sim \log(T)$ as expected. Moreover, we note that $f_{\mathrm{CUE}}(2) = 1/12$, and it can be shown that $a(2) = 6/\pi^2$, thereby corroborating the Ingham result.

Thus, if proven to be true, this correspondence reveals a fundamental relationship between the statistical properties of CUE matrices and the zeta function along the critical line. It is also unique due to its analytically supported nature, as much of zeta-RMT study is conjectural and numerical in nature, as will be demonstrated in the following section.

6 Pair Correlation and Zero Spacing

We now analyze the pair correlations of the non-trivial zeros on $s = \frac{1}{2} + it$, which in this section will be referred to simply as zeros.

The pair correlation function $R_2(r)$ describes the frequency of occurrence of a pair of two zeros separated by distance r on critical line [11]. If each pair with distance r is equally likely to occur, that is, the zeros are uniformly distributed, then $R_2(r) = 1$. As we will see with the zeros and eigenvalues, however, the frequency becomes extremely low as $r \to 0$ compared to longer distances, resulting in repulsion between zeros.

The counts are normalized by dividing them by the mean distance between each adjacent zero on the critical line. This is also done with eigenvalue pairs so that each pair correlation is measured on the same scale.

6.1 Contour Integration Argument

We now investigate the exact form of the pair correlation for the zeros by outlining a simplified form of a contour integration argument which uses the logarithmic derivative of $\zeta(s)$, given as $\zeta'(s)/\zeta(s)$. Observe that this logarithmic derivative has a simple pole at every non-trivial zero $z_n = 1/2 + i\gamma_n$ with residue $+1$, as can be trivially computed with Ablowitz 4.4.2 [1].

Let C be a rectangular closed contour whose vertices are $\varepsilon \pm iT$ and $\sigma \pm iT$ for a small $\varepsilon > 0$. For geometric simplicity, we choose $\sigma = 1 + \varepsilon$ so that our contour effectively encloses the critical strip but is offset to the right by ε to avoid ambiguities such as integrating along the $s = 1$ pole.

We are interested in the result as $T \to \infty$. Assuming the Riemann Hypothesis, the only zeros of the zeta function enclosed by this contour are those along the critical line. We also enclose the simple pole at $s = 1$.

We now integrate this logarithmic derivative over a closed contour with a test function g, applying the argument principle [1] (Ablowitz 4.4.5) to get:

$$\frac{1}{2\pi i} \oint_C g(s) \frac{\zeta'(s)}{\zeta(s)} ds = \sum_{i=1}^{M_z} g(z_i) - g(1) \tag{26}$$

where z_i are the non-trivial zeros inside of our contour and M_z is their count. The $g(1)$ term arises from the one simple pole.

H.L. Montgomery (1973) performs this integration procedure with a test function chosen such that the integrals along the top and bottom sides of the rectangle contribute $O(T^{-1} \log^2 T)$ and thus vanish as $T \to \infty$, leaving the integrals along the left and right sides. The left side's contribution is a manageable computation for the correct choice of test function and the right side is solved for using the von Mangoldt Dirichlet series, which yields a prime sum that is handled using prime number theory [3, pp. 104-110].

6.2 Montgomery Pair Correlation Conjecture

The result is an explicit asymptotic expression for the sum of g over the zeros, which Montgomery uses in conjuction with Fourier spectrum analysis to arrive at his pair correlation conjecture [11, p. 184]:

$$R_2(r) = 1 - \left(\frac{\sin(\pi r)}{\pi r}\right)^2 \tag{27}$$

This remains a conjecture rather than a universally proven theorem due to a necessary constraint made on the test function. As expected, R_2 approaches a limit of 0 for $r \to 0$, validating the repulsion behavior.

The significance of this result lies in the fact that this pair correlation function is identical to that of the eigenvalues of random matrices from the Gaussian and Circular unitary ensembles [9, p. 196]. This correspondence was noticed in 1972 by Freeman Dyson, who studied the Gaussian and Circular ensembles extensively [4].

6.3 Numerical Results

Equation 27 suggests that the zeros on the critical line are repulsive, or that two zeros are unlikely to be close (or have $r \approx 0$):

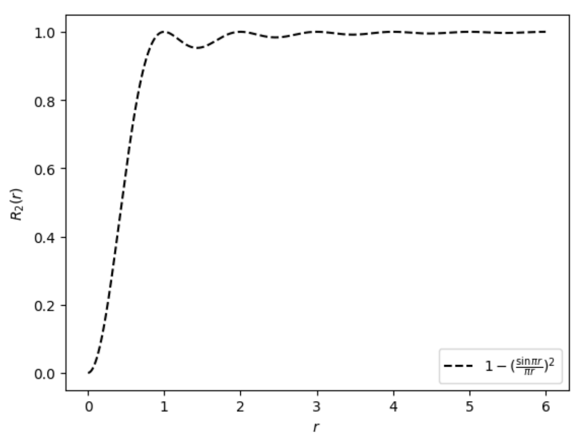


Figure 2: Pair correlation $R_2\left(r\right)$ for the non-trivial zeros assuming the Riemann Hypothesis.

Figure 2 shows that when $r \to 0$, the count of these pairs rapidly decreases to 0. When pairs have $r \gg 0$, they are predicted to have approximately the same count, so two points tend to repel each other when they get close.

The pair correlation can be computed numerically as follows:

1. For each pair of points p_j, p_k, find $|p_j - p_k|$.

2. Normalize the distances, dividing by the mean distance between two adjacent points: $r = \frac{|p_j - p_k|}{\text{mean}(|p_j - p_{j+1}|)}$.

3. Approximate count of each r with a histogram.

4. Normalize histogram by dividing each bin by mean count on interval for r.

5. For the CUE matrix, use $p_j = \theta_j$.

Code for generating GUE and CUE matrices are found in the Appendix and procedures for their construction are given by [6, p. 5] and [10, p. 11] respectively. Following the procedure above including normalization, we see that indeed the zero and eigenvalue statistics appear to match, although noisily due to poor approximation of $Z(t)$ with our simple Riemann-Siegel implementation:

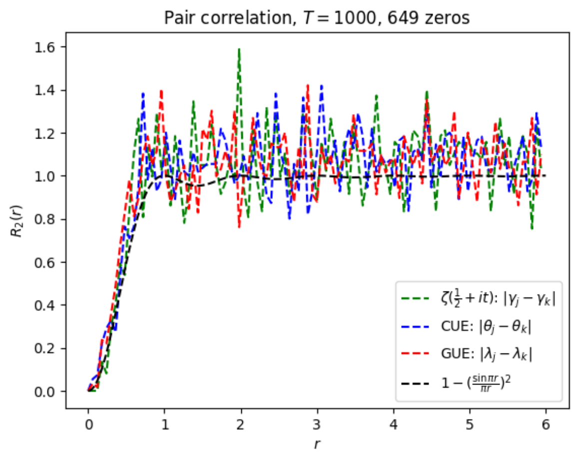


Figure 3: Normalized pair correlation for the first 649 ζ zeros, GUE and CUE eigenvalues for $N \times N$, $N = 1000$ matrices.

Figure 3 shows that the Zeta zero and RMT eigenvalue pair correlations all exhibit the same repulsive behavior when $r \to 0$. This corroborates the suggested connection between random matrix statistics and the zeta function.

7 Conclusion

In this report, we have explored the statistical correspondence between the non-trivial zeros of the Riemann zeta function and the eigenvalue distributions of random matrices. While the Gaussian Unitary Ensemble historically laid the groundwork for this connection, transitioning to the Circular Unitary Ensemble has proven to uncover new insights into the relationship between spectral statistics of random matrices and zeta zeros. We have sought to convince readers of this connection in two ways: the moments correspondence given by the Keating-Snaith conjecture, and the striking agreement between the pair correlations of zeta zeros and that of the GUE and CUE eigenvalues.

These results gain significance within the context of the Riemann Hypothesis when reconsidering the Hilbert-Pólya conjecture, which suggests that the Riemann Hypothesis can be proven true if one can show that there is a Hermitian operator with eigenvalues that correspond to the non-trivial zeros [12, p. 274]. The GUE is made up of Hermitian matrices, and we have shown that their eigenvalue distributions seem to reproduce the pair correlation statistics of the zeros. However, this is a statistical correspondence rather than a direct match between eigenvalues and zeros, meaning the GUE fails to satisfy the conjecture. We have demonstrated the same pair correlation correspondence for CUE matrices, and in fact show the CUE going further with analytically verified predictions of zeta function moments. This result is reached despite the CUE lacking a Hermitian property, suggesting that while the Hilbert-Pólya conjecture is central to the RMT-zeta link, approaches towards the Riemann Hypothesis applying RMT should not restrict themselves to solely Hermitian matrix classes lest they neglect the compelling correspondences involving CUE matrices.

For the future, it may be worthwhile to examine other random matrix ensembles such as those related to the orthogonal or symplectic groups to understand why they may or may not experience the same correspondences as we demonstrated for the GUE and CUE. We also consider exploring additional statistical properties such as distributions of spacings between adjacent zeros and higher-order correlations to solidify the notion that there is a deeper underlying relationship between random matrix theory and the Riemann zeta function, one that has yet to be fully uncovered, to potentially solve a centuries-old problem.

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A Code

Below is our Python code for the numerical work in this report.

```
import numpy as np
import mpmath as mp
from scipy.special import loggamma, zeta
from scipy.integrate import quad
import math
import matplotlib.pyplot as plt
import matplotlib as mpl
from matplotlib.ticker import MaxNLocator

def theta(t):
    return np.imag(loggamma(0.25 + 0.5j*t)) - 0.5*t*np.log(np.pi)

def Z(t):
    # Z computed from Zeta function
    return np.real(np.exp(1j*theta(t))*zeta(0.5 + 1j*t))

def Z_hat(t):
    # Z estimated from Riemann-Siegel formula
    tau = t/(2*np.pi)
    tausqrt = np.sqrt(tau)
    m = int(np.floor(np.real(tausqrt)))
    z = 2*(tausqrt-m)-1
    def f(n):
        return (1/np.sqrt(n))*np.cos(theta(t) - t*np.log(n))
    def phi(z):
        return np.cos(np.pi*(4*z**2+3)/8)/np.cos(np.pi*z)
    return 2*sum(f(n) for n in range(1, m+1)) + (-1)**(m+1)*(1/(tau**0.25))*(phi
        (z))

def bisection(f, a, b, tol=1e-6, max=1000):
    # bisection method for root-finding
    fa = f(a)
    fb = f(b)
    fab = fa*fb
    if fab < 0:
        mid = (a+b)/2
        fmid = f(mid)
        if (abs(fmid) <= tol or max <= 0):
            return mid
        if (fmid*fb < 0):
            return bisection(f, mid, b, max=max-1)
        elif (fa*fmid < 0):
            return bisection(f, a, mid, max=max-1)
    return None

def roots(I, f=Z_hat):
    # I is interval, f is root-finder, apply f at sign changes
    zeros = []
    for i in range(1, len(I)):
        eval = bisection(f, I[i-1], I[i])
        if eval is not None:
            zeros.append(eval)
```

```
    return zeros

def GUE(n):
    # Generate a GUE matrix
    A = np.random.normal(size=(n, n)) + 1j*np.random.normal(size=(n, n))
    return (A + A.conj().T) / 2

def CUE(n):
    # Generate a CUE matrix
    Z = np.random.normal(size=(n, n)) + 1j*np.random.normal(size=(n, n))
    Q, R = np.linalg.qr(Z)
    diagonals = np.diag(R)
    return Q * (diagonals / np.abs(diagonals))

def pair_corr(points, norm=1):
    # Pair correlation - points is array of zeros/eigenvalues
    pair_corrs = []
    n = len(points)
    for i in range(1, n):
        for j in range(0, i):
            pair_corrs.append(np.abs(points[i]-points[j]))
    return np.array(pair_corrs) / norm

# Generate interval of t=1-100
I = np.linspace(1, 100, 1000)

# Riemann-Seigel Z outputs on x=[1, 100]
Z_hats = [Z_hat(t) for t in I]

# Find zeros on I
zeros = roots(I)

# Plot of zeros up to T=100
plt.plot(I, Z_hats, c='red', zorder=1)
plt.scatter(zeros, [0]*len(zeros), s=15, c='green', zorder=3, linewidths=1,
   edgecolors='black')
plt.axhline(0, color='black', linestyle='--', zorder=2, linewidth=1)
plt.xlabel(r'$t$')
plt.ylabel(r'$Z(t)$')

# New interval of t=1-1000
I1=np.linspace(1, 1000, 5000)

# 649 zeros up to T=1000
zeros_I1 = roots(I1)

# Create Zeta zero pair correlation
pc_zeros_I1 = pair_corr(zeros_I1, np.mean(np.diff(zeros_I1)))

# Sample GUE eigenvalues
eigvals_I1 = np.real(np.linalg.eigvals(GUE(1000)))

# Sample CUE eigenangles
cmp_eigs_I1 = np.linalg.eigvals(CUE(1000))
eig_angles_I1 = np.imag(np.log(cmp_eigs_I1))
```

```
# Pair correlation for sampled GUE and CUE eigenvalues
pc_eigs_I1 = pair_corr(eigvals_I1, np.mean(np.diff(np.sort(eigvals_I1))))
pc_angles_I1 = pair_corr(eig_angles_I1, np.mean(np.diff(np.sort(eig_angles_I1))))

# Plot normalized pair correlations
max_u = 6
hist, bins = np.histogram(pc_zeros_I1, range=(0, max_u), bins=100, density=True)
hist1, bins1 = np.histogram(pc_angles_I1, range=(0, max_u), bins=100, density=True)
hist2, bins2 = np.histogram(pc_eigs_I1, range=(0, max_u), bins=100, density=True)
plt.plot(bins[:-1], max_u*hist, color='green', linestyle='--', label=r'$\zeta(\frac{1}{2}+it)$: $|\gamma_j - \gamma_k|$')
plt.plot(bins1[:-1], max_u*hist1, color='blue', linestyle='--', label=r'CUE: $|\theta_j - \theta_k|$')
plt.plot(bins2[:-1], max_u*hist2, color='red', linestyle='--', label=r'GUE: $|\lambda_j - \lambda_k|$')
u = np.linspace(0.001, max_u, 1000)
plt.plot(u, (1-(np.sin(np.pi*u)/(np.pi*u))**2), color='black', linestyle='--', zorder=4, label=r'$1-(\frac{\sin \pi r}{\pi r})^2$')
plt.xlabel(r'$r$')
plt.ylabel(r'$R_2(r)$')
plt.title(f'Pair correlation, $T=${1000}, {len(zeros_I1)} zeros')
plt.legend()
```